

Title	\$G_2\$-Geometry of Overdetermined Systems of Second Order (Geometry of Submanifolds : Elie Cartan and the 21st Century)
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Citation	数理解析研究所講究録 (2001), 1206: 95-106
Issue Date	2001-05
URL	<a href="http://hdl.handle.net/2433/41033">http://hdl.handle.net/2433/41033</a>
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Type	Departmental Bulletin Paper
Textversion	publisher

## $G_2$ -Geometry of Overdetermined Systems of Second Order

By

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### Introduction.

Discovery of E. Cartan in

[C1] *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Ec. Normale. 27 (1910), 109-192

Overdetermined ( involutive) system :

$$(A) \quad \frac{\partial^2 z}{\partial x^2} = \frac{1}{3} \left( \frac{\partial^2 z}{\partial y^2} \right)^3, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2} \left( \frac{\partial^2 z}{\partial y^2} \right)^2.$$

Single equation of Goursat type:

$$(B) \quad 9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0,$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

14-dimensional Exceptional Simple Lie Algebra  $G_2$

### The Plan of This TALK

Main Theme      **Contact Geometry of Second Order**

- Grassmannian Construction of Jet Spaces (§1)

$\implies PD$ -manifolds (§4).

- Geometry of Linear Differential Systems( Tanaka Theory)(§2)

$\implies$  Differential Systems associated with SGLA (Simple Graded Lie Algebras) (§3).

- Link between them

$\implies$  Reduction Theorems for  $PD$  manifolds (§4, §5)

Together combined to discuss

$G_2$ -Geometry of Overdetermined Systems of Second Order (§6)

### §1.Second Order Contact Manifolds.

GrassmannBundle:

$M$ : a manifold of dimension  $m + n$

$$J(M, n) = \bigcup_{x \in M} J_x, \quad J_x = \text{Gr}(T_x(M), n).$$

Canonical System  $C$  on  $J(M, n)$  :

$\forall u \in J(M, n)$

$$C(u) = \pi_*^{-1}(u) \subset T_u(J(M, n)).$$

Inhomogeneous Grassmann coordinate:  $x_o = \pi(u_o) \in U'$ ;  $(x_1, \dots, x_n, z^1, \dots, z^m)$

$$U = \{u \in \pi^{-1}(U') \mid \pi(u) = x \in U' \text{ and } dx_1 \wedge \dots \wedge dx_n|_u \neq 0\};$$

Coordinates  $(x_1, \dots, x_n, z^1, \dots, z^m, p_1^1, \dots, p_n^m)$  are introduced by

$$dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u.$$

On a canonical coordinate system  $(x_1, \dots, x_n, z^1, \dots, z^m, p_1^1, \dots, p_n^m)$

$$C = \{\varpi^1 = \dots = \varpi^m = 0\},$$

where  $\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha dx_i$ .

$m = 1 \implies$  Contact Manifold

$$\varphi : M \rightarrow \hat{M} : \text{diffeomorphism} \implies \varphi_* : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$$

**Theorem 2.1 (Bäcklund).**  $M, \hat{M}$  : manifolds of dimension  $m + n$ .

Assume  $m \geq 2$ .  $\Phi : (J(M, n), C) \rightarrow (J(\hat{M}, n), \hat{C})$  ; isomorphism

$\implies \exists_1 \varphi : M \rightarrow \hat{M}$  such that  $\Phi = \varphi_*$ .

$$(J, C) : \text{Contact Manifold} \implies (L(J), E)$$

Lagrange-Grassmann Bundle

$$L(J) = \bigcup_{u \in J} L_u \xrightarrow{\pi} J$$

$$L_u = \{\text{Legendrian subspaces of } (C(u), d\varpi)\}.$$

$\forall v \in L(J)$

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J)$$

where  $(x_1, \dots, x_n, z, p_1, \dots, p_n, p_{11}, \dots, p_{nn})$  and  $p_{ij} = p_{ji}$

$$E = \{\varpi = \varpi_1 = \dots = \varpi_n = 0\},$$

where

$$\begin{cases} \varpi = dz - \sum_{i=1}^n p_i dx_i, \\ \varpi_i = dp_i - \sum_{j=1}^n p_{ij} dx_j \end{cases}$$

**Theorem 2.2.**  $\Phi : (L(J), E) \rightarrow (L(\hat{J}), \hat{E})$ ; isomorphism  $\implies \exists_1 \varphi : (J, C) \rightarrow (\hat{J}, \hat{C})$  such that  $\Phi = \varphi_*$ .

## §2. Geometry of Linear Differential Systems (Tanaka Theory).

$M$ : a manifold of dimension  $d$

$D \subset T(M)$ : subbundle of rank  $r$  ( $s + r = d$ )

$$D = \{\omega_1 = \dots = \omega_s = 0\}.$$

$(M, D)$ : completely integrable

$$\iff D = \{dx_1 = \dots = dx_s = 0\}$$

$$\iff d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \quad (1 \leq i \leq s)$$

$$\iff [D, D] \subset D \text{ where } D = \Gamma(D)$$

**Derived System**  $\partial D$ :  $\partial D = D + [D, D]$ .

**Cauchy Characteristic System**  $Ch(D)$ :

$$Ch(D)(x) = \{X \in D(x) \mid X \lrcorner d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \text{ for } i = 1, \dots, s\},$$

$k$ -th Derived System  $\partial^k D$  :

$$\partial^k D = \partial(\partial^{k-1} D)$$

$k$ -th Weak Derived System  $\partial^{(k)} D$  :

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

### Symbol Algebras

$(M, D) : \text{regular}$

(S1)  $\exists \mu > 0$  such that, for all  $k \geq \mu$ ,

$$D^{-k} = \dots = D^{-\mu} \supsetneq \dots \supsetneq D^{-2} \supsetneq D^{-1} = D,$$

(S2)  $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$  for all  $p, q < 0$ .

$(M, D) : \text{regular such that } T(M) = D^{-\mu}.$

$\forall x \in M,$

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

$$\mathfrak{g}_{-1}(x) = D^{-1}(x), \mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$$

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where  $\tilde{X} \in \Gamma(D^p), X = \varpi_p(\tilde{X}_x) \in \mathfrak{g}_p(x), \tilde{Y} \in \Gamma(D^q), Y = \varpi_q(\tilde{Y}_x) \in \mathfrak{g}_q(x).$

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \quad \text{for } p < -1.$$

Conversely given a

### Fundamental Graded Lie Algebra :

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

i.e., Nilpotent GLA satisfying the generating condition :  $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$  for  $p < -1$

$\implies$

$(M(\mathfrak{m}), D_{\mathfrak{m}}) : \text{Standard Differential System of type } \mathfrak{m}$

$\implies$

$\mathfrak{g}(\mathfrak{m}) : \text{Prolongation of } \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$

Our Problem :

When does  $\mathfrak{g}(\mathfrak{m})$  become finite dimensional and simple ?

### Symbol Algebra of $(L(J), E) :$

$$E = \{\varpi = \varpi_1 = \dots = \varpi_n = 0\},$$

$$\mathfrak{c}^2(n) = \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where  $\mathfrak{c}_{-3} = W, \mathfrak{c}_{-2} = W \otimes V^*,$

$$\mathfrak{c}_{-1} = V \oplus W \otimes S^2(V^*).$$

Coframe:

Dual frame:

$$\{\varpi, \varpi_i, dx_i, dp_{ij}\}, \quad \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial p_i}, \frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}} \right\}$$

where

$$\begin{aligned} \frac{d}{dx_i} &= \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{j=1}^n p_{ij} \frac{\partial}{\partial p_j} \\ \left[ \frac{\partial}{\partial p_{ij}}, \frac{d}{dx_k} \right] &= \delta_k^i \frac{\partial}{\partial p_j} + \delta_k^j \frac{\partial}{\partial p_i}, \\ \left[ \frac{\partial}{\partial p_i}, \frac{d}{dx_k} \right] &= \delta_k^i \frac{\partial}{\partial z}. \end{aligned}$$

$$\partial E = \{\varpi = 0\} = \pi_*^{-1}C, \quad Ch(\partial E) = \text{Ker} \pi_*.$$

### §3. Differential Systems associated with Simple Graded Lie Algebras.

Gradation of  $\mathfrak{g}$

$\mathfrak{g}$ : Simple Lie Algebra over  $\mathbb{C}$

$\mathfrak{h}$ : Cartan Subalg. ;  $\Phi \subset \mathfrak{h}^*$ : Root System

$\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ : Simple Root System

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha},$$

$$\Delta_1 \subset \Delta: \text{Fix}, \quad \Phi^+ = \bigcup_{p \geq 0} \Phi_p^+,$$

$$\Phi_p^+ = \left\{ \alpha = \sum_{i=1}^{\ell} n_i \alpha_i \mid \sum_{\alpha_i \in \Delta_1} n_i = p \right\},$$

$$\begin{cases} \mathfrak{g}_p = \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_\alpha, & (p > 0) \\ \mathfrak{g}_0 = \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_{-\alpha}, \\ \mathfrak{g}_{-p} = \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_{-\alpha}, \end{cases}$$

$$\text{Then } [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \quad \text{for } p, q \in \mathbb{Z}.$$

$$\text{Generating Condition: } \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p \quad (*) \quad \mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \quad \text{for } p < -1$$

$$\Delta_1 \subset \Delta \implies (X_\ell, \Delta_1): \quad \mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$$

$$\text{where } \mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta), \quad \theta = \sum_{i=1}^{\ell} n_i(\theta) \alpha_i$$

**Theorem 4.1.**  $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ : Simple Graded Lie Algebra over  $\mathbb{C}$  satisfying  $(\star)$ .  
 $X_\ell$ : Dynkin Diagram of  $\mathfrak{g} \implies \exists_1 \Delta_1 \subset \Delta$  s.t.  $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p \cong (X_\ell, \Delta_1)$

Classification of  $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$  with  $(\star)$

$$\iff \text{Classification of Parabolic subalgebras } \mathfrak{g}' = \bigoplus_{p \geq 0} \mathfrak{g}_p$$

$$(X_\ell, \Delta_1) \implies M_{\mathfrak{g}} = G/G': \text{ R-space} \quad \mu \geq 2 \quad \mathfrak{g}_{-1} \implies D_{\mathfrak{g}} \quad \text{on } M_{\mathfrak{g}}$$

$$(M_{\mathfrak{g}}, D_{\mathfrak{g}}) \supset (M_{\mathfrak{m}}, D_{\mathfrak{m}}), \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p.$$

**Theorem 4.2.**  $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ : Simple Graded Lie Algebra over  $\mathbb{C}$  satisfying  $(\star)$ .  
 Except for (1), (2), (3),

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p \cong \mathfrak{g}(\mathfrak{m}),$$

where  $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ .

- (1)  $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  is of depth 1 ( $\mu = 1$ ).
- (2)  $\mathfrak{g} = \bigoplus_{p=-2}^2 \mathfrak{g}_p$  is a contact gradation.
- (3)  $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$  is isomorphic with  $(A_\ell, \{\alpha_1, \alpha_i\})$  ( $1 < i < \ell$ )  $(C_\ell, \{\alpha_1, \alpha_\ell\})$ .

Corresponding R-spaces

- (1)  $\implies$  Compact Hermitian Symmetric Spaces
- (2)  $\implies$  Standard Contact Manifolds
- (3)  $(A_\ell, \{\alpha_1, \alpha_i\}) \implies (J(\mathbb{P}^\ell, i-1), C)$   $(C_\ell, \{\alpha_1, \alpha_\ell\}) \implies (L(\mathbb{P}^{2\ell-1}), E)$ .

#### §4. Geometry of PD-manifolds.

$R \subset L(J)$ : submanifold satisfying

$$(R.0) \quad p: R \rightarrow J; \text{ submersion,}$$

$$\text{On } L(J), \quad C^1 = \partial E, \quad C^2 = E$$

$$\text{On } R, \quad D^1 = C^1|_R, \quad D^2 = C^2|_R$$

$(R; D^1, D^2)$  satisfies :

$$(R.1) \quad D^1: \text{codim. } 1, \quad D^2: \text{codim. } n+1,$$

$$(R.2) \quad \partial D^2 \subset D^1,$$

$$(R.3) \quad Ch(D^1) \subset D^2: \text{codim. } n,$$

$$(R.4) \quad Ch(D^1) \cap Ch(D^2) = \{0\}.$$

**Triplet**  $(R; D^1, D^2) : \text{PD-manifold} \iff (R.1) \sim (R.4)$

$$\{0\} = Ch(D^1) \cap Ch(D^2) \subset Ch(D^1) \subset D^2 \subset \partial D^2 \subset D^1 \subset T(R)$$

### Realization Theorem for PD-manifold

$$(i) \quad (R.1) \text{ and } (R.3) \implies (J, C)$$

$$J = R/Ch(D^1), \quad D^1 = p_*^{-1}(C),$$

where  $p : R \rightarrow J = R/Ch(D^1)$ .

$$(ii) \quad (R.1) \text{ and } (R.2) \implies \iota(v) = p_*(D^2(v)) \subset C(u) : \text{Legendrian}$$

$$(iii) \quad (R.4) \implies \iota : R \rightarrow L(J) : \text{immersion}$$

### Theorem 5.1.

$$\Phi : (R; D^1, D^2) \rightarrow (\hat{R}; \hat{D}^1, \hat{D}^2) : \text{isomorphism} \implies$$

$$\exists_1 \varphi : (J, C) \rightarrow (\hat{J}, \hat{C}) : \text{contact diffeo s.t.};$$

$$\begin{array}{ccc} R & \xrightarrow{\iota} & L(J) \\ \Phi \downarrow & & \downarrow \varphi_* \\ \hat{R} & \xrightarrow{\hat{\iota}} & L(\hat{J}). \end{array}$$

Compatibility Condition (C)

$$(C) \quad p^{(1)} : R^{(1)} \rightarrow R \text{ is onto.}$$

where  $R^{(1)}$ : the first prolongation of  $(R; D^1, D^2)$ .

**Theorem 5.2.**  $(R; D^1, D^2) : \text{PD-manifold satisfying the condition (C)}.$

$\forall v \in R:$

$$\dim D^1(v) - \dim \partial D^2(v) = \dim Ch(D^2)(v).$$

*Especially*  $D^1 = \partial D^2 \iff Ch(D^2) = \{0\}.$

In case  $\text{rank } Ch(D^2) > 0,$

$$\text{Geometry of } (R; D^1, D^2) \implies \text{Geometry of } (X, D),$$

where  $X = R/Ch(D^2), D^2 = \rho_*^{-1}(D), \rho : R \rightarrow X.$



### §5. Single Equations of Goursat Type.

$L(J) \supset R = \{F(x_i, z, p_i, p_{ij}) = 0\}$ : Hypersurface s.t.  $p : R \rightarrow J$  ; submersion,

$R$  is of (weak) parabolic type at each  $v \in R$

$$\iff \left( \frac{\partial F}{\partial p_{ij}}(v) \right) : \text{rank } 1 \text{ at each } v \in R$$

$$\iff (R, D^2) : \text{regular of type } \mathfrak{s}:$$

$$\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1},$$

where  $\mathfrak{s}_{-3} = \mathbb{R}$ ,  $\mathfrak{s}_{-2} = V^*$ ,  $\mathfrak{s}_{-1} = V \oplus \mathfrak{f} \mathfrak{f} \subset S^2(V^*)$ ;  $(\mathfrak{f})^\perp = \langle e^2 \rangle \subset S^2(V)$ ,  $e \in V$ .

$\iff \exists$  Coframe  $\{\varpi, \varpi_a, \omega_a, \varpi_{1\alpha}, \varpi_{\alpha\beta}\}$  ( $1 \leq a \leq n, 2 \leq \alpha \leq \beta \leq n$ ) on  $R$  such that

$$D^2 = \{\varpi = \varpi_1 = \dots = \varpi_n = 0\},$$

$$\begin{cases} d\varpi \equiv \omega_1 \wedge \varpi_1 + \dots + \omega_n \wedge \varpi_n \pmod{\varpi}, \\ d\varpi_1 \equiv \omega_2 \wedge \varpi_{12} + \dots + \omega_n \wedge \varpi_{1n} \pmod{\varpi, \varpi_1, \dots, \varpi_n}, \\ d\varpi_\alpha \equiv \omega_1 \wedge \varpi_{\alpha 1} + \dots + \omega_n \wedge \varpi_{\alpha n} \pmod{\varpi, \varpi_1, \dots, \varpi_n}. \end{cases}$$

where  $\varpi_{\alpha\beta} = \varpi_{\beta\alpha}$   $\varpi_{1\alpha} = \varpi_{\alpha 1}$ ,  $2 \leq \alpha, \beta \leq n$ .

$R$  is a equation of **Goursat type**

$\iff R$  ; (weak) parabolic type s.t.  $M(E)$ ; completely integrable, where  $M(E)$  is the **Monge system** ;

$$M(E) = \{\varpi = \varpi_1 = \dots = \varpi_n = \omega_\alpha = \varpi_{1\alpha} = 0 \quad (2 \leq \alpha \leq n)\}.$$

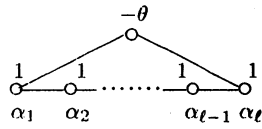
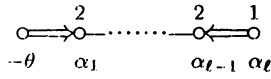
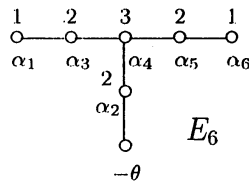
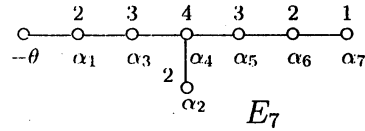
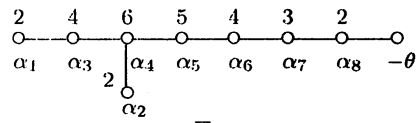
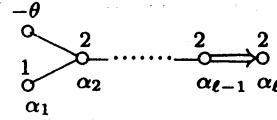
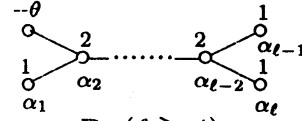
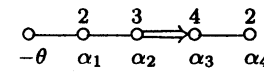
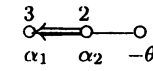
The First Order Covariant System  $N(E)$

$$N = N(E) = \{\varpi = \varpi_1 = 0\}.$$

By Two Step Reductions

**Geometry of  $(R, D^2)$ ; Goursat Type  $\implies$  Geometry of  $(Y, D_N)$ ; Type  $c^1(n-1, 2)$ ,**

where  $Y = R/Ch(N)$ ,  $N = \rho_*^{-1}(D_N)$ ,  $\rho : R \rightarrow Y$ .

 $A_\ell (\ell \geq 2)$  $C_\ell (\ell \geq 2)$  $E_6$  $E_7$  $E_8$  $B_\ell (\ell \geq 3)$  $D_\ell (\ell \geq 4)$  $F_4$  $G_2$ 

Extended Dynkin Diagrams with the coefficient of the highest root

## §6. $G_2$ -geometry.

### 6.1. Standard Contact Manifolds

$\mathfrak{g}$ : Simple Lie Algebra over  $\mathbb{C}$

$\theta$ : Highest Root

$(X_\ell, \Delta_\theta)$ : Contact Gradation  $\implies \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$

$(J_\mathfrak{g}, C_\mathfrak{g})$ : Standard Contact Manifolds  $\Leftarrow$  Boothby

[Projectiviation of the (co-)ajoint orbit through the highest root vector]

$\Delta_\theta \iff$  **Extended Dynkin Diagram**

## 6.2. Gradation of $G_2$ .

$$\begin{aligned} \odot_{\alpha_1} &\Leftarrow \odot_{\alpha_2}, & \theta &= 3\alpha_1 + 2\alpha_2. \\ \Delta_1 &\subset \Delta = \{\alpha_1, \alpha_2\} \end{aligned}$$

$$(G1) \quad \Delta_1 = \{\alpha_1\}. \quad \mu = 3,$$

$$\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

where  $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$ ,  $\dim \mathfrak{g}_{-2} = 1$ .

$$(G2) \quad \Delta_1 = \{\alpha_2\}. \quad \mu = 2$$

$$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} : \text{Contact Gradation}$$

$$(G3) \quad \Delta_1 = \{\alpha_1, \alpha_2\}. \quad \mu = 5,$$

$$\mathfrak{m} = \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

where  $\dim \mathfrak{g}_{-1} = 2$  and  $\dim \mathfrak{g}_p = 1$  for others.

**Root System  $G_2$**

$$\text{rank } \mathfrak{g} = \dim \mathfrak{h} = 2$$

$\Delta = \{\alpha_1, \alpha_2\}$ : **Simple Root System**

$\Phi^+$  consists of the following roots

$$\alpha_1, \alpha_2,$$

$$\alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1,$$

$$2\alpha_2 + 3\alpha_1$$

**Type  $G_2$**

$(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ : **Standard Contact Manifold**

$$\dim J_{\mathfrak{g}} = 5$$

$L(J_{\mathfrak{g}})$  : **Lagrange-Grassmann Bundle**

$$\dim L(J_{\mathfrak{g}}) = 8$$

**Orbits Decomposition**

$$L(J_{\mathfrak{g}}) = O \cup R_1 \cup R_2,$$

(1)  $O$ : Open orbit,

(2)  $R_1$ : Codim 1, the Global Model of  $(B)$ ,

(3)  $R_2$ : Codim 2, the Global Model of  $(A)$ .

$$R_2: \text{compact} \quad \Rightarrow \quad (G_2, \{\alpha_1, \alpha_2\})$$

$$X_{\ell} \not\cong A_{\ell} \quad \Rightarrow \quad \Delta_{\theta} = \{\alpha_{\theta}\}$$

For **Exceptional Simple Lie Algebras**,  $\exists_1 \quad \alpha_G : \mathbf{3}$  next to  $\alpha_{\theta}$

$$(X_{\ell}, \{\alpha_G\}): \mu = 3 \quad \Longleftrightarrow \quad (M_{\mathfrak{g}}, D_{\mathfrak{g}})$$

$$\mathfrak{g}_{-3} = W, \quad \mathfrak{g}_{-2} = V \quad \mathfrak{g}_{-1} = W \otimes V^*.$$

$$\dim \mathfrak{g}_{-3} = 2$$

i.e.,  $(M_{\mathfrak{g}}, \partial D_{\mathfrak{g}})$ : regular of type  $c^1(r, 2)$ .

$$(J_{\mathfrak{g}}, C_{\mathfrak{g}}) \quad \Longleftrightarrow \quad (X_{\ell}, \{\alpha_{\theta}\})$$

$$\uparrow$$

$$L(J_{\mathfrak{g}}) \supset R_2 \quad \Rightarrow \quad (X_{\ell}, \{\alpha_{\theta}, \alpha_G\})$$

$$\downarrow$$

$$(X, D) \quad \Longleftrightarrow \quad (X_{\ell}, \{\alpha_G\})$$

$$L(J_{\mathfrak{g}}) \supset R_1 \quad \Leftarrow \quad (M_{\mathfrak{g}}, \partial D_{\mathfrak{g}})$$

## References

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